

Scaling of high-wavenumber energy spectra in the unit aspect-ratio rotating Boussinesq system

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Phenomenological and numerical studies of the small scale spectra of energy are presented for high Reynolds number rotating Boussinesq flows in unit aspect-ratio domains. We introduce a non-dimensional parameter $\Gamma(\mathbf{k}) = \frac{fk_z}{Nk_h}$, where the wavevector \mathbf{k} has vertical component k_z and horizontal component k_h , f is the Coriolis frequency and N is the Brunt-Väisälä frequency. While requiring that f and N are such that the potential vorticity is nearly linear in the dynamical variables, we deduce that for $\Gamma \ll 1$, the potential enstrophy suppresses the transfer of horizontal kinetic energy into large k_h modes while forcing it to become independent of k_z , scaling as k_h^{-5} . For $\Gamma \gg 1$, the potential enstrophy suppresses the transfer of potential energy into the large k_z modes while forcing it to become independent of k_h , scaling as k_z^{-5} . The usual scaling exponent -3 is adjusted down to -5 based on a heuristic argument which links the downscale fluxes of potential enstrophy and energy by their ratio. Spectra computed from high-resolution simulations of the Boussinesq equations with isotropic low-wavenumber forcing are used to explore such anisotropic constraints on the energy and provide *a posteriori* justification for the joint flux ansatz used to obtain the -5 scaling exponent. In simulations for which $f/N \ll 1$, the asymptotic regime of $\Gamma \ll 1$ is achieved and the horizontal kinetic

energy is shown to become independent of k_z and scale as k_h^{-5} . This collapse results in a uniform k_z^0 scaling for all k_h and hence a uniform k^0 scaling of the energy in the spherical wavenumber, indicating extremely efficient net downscale transfer of energy. In simulations for which $f/N \gg 1$, the trend towards the asymptotic regime of $\Gamma \gg 1$ is observed although the potential energy does not become entirely independent of k_h as predicted. In simulations with $f/N = 1$ where rotation and stratification are equally strong, both $\Gamma \ll 1$ and $\Gamma \gg 1$ regimes may be recovered depending on the aspect-ratio of the wavevector, $\tau = k_z/k_h$, which is also the cotangent of the wavevector angle to the vertical. The dependence of the spectra on τ results, on average, in shallower scaling exponent values between -1 and $-5/3$ in both k_h and k_z implying an effectively more isotropic downscale distribution of energy in this limit. In all cases the empirical evidence points to both energy and potential enstrophy being jointly transferred downscale with the spectral scaling of the the former constrained by the latter.

1. Introduction

The Boussinesq approximation for hydrodynamic flow is the cornerstone for many applications in which a suitable model for scalar transport is needed. It is physically realistic in that it postulates that the fluctuations of the scalar are small compared to its background value and is computationally tenable for linear stratification profiles and simple boundary conditions. The Boussinesq approximation in the rotating frame forms the parent system for the derivation of further simplifications such as the shallow-water, hydrostatic and quasi-geostrophic (QG) models which are standard approximations in geophysical regimes which include rotation and stratification of the flow.

The Boussinesq system for a stably stratified flow in a rotating frame is given by:

$$\frac{D}{Dt}\mathbf{u} + f\hat{\mathbf{z}} \times \mathbf{u} + \nabla p + N\theta\hat{\mathbf{z}} = \nu\nabla^2\mathbf{u} + \mathcal{F} \quad (1.1)$$

$$\frac{D}{Dt}\theta - Nw = \kappa\nabla^2\theta \quad (1.2)$$

$$\nabla \cdot \mathbf{u} = 0,$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla, \quad (1.3)$$

where \mathbf{u} is the velocity, w is its vertical component, p is the effective pressure and \mathcal{F} is an external input or force. The total density is given by $\rho_T(\mathbf{x}) = \rho_0 - bz + \rho(\mathbf{x})$, such that $|\rho| \ll |bz| \ll \rho_0$ where ρ_0 is the constant background, b is constant and larger than zero for stable stratification in the vertical z -coordinate, ρ is the density fluctuation. The density is normalized to $\theta = \rho(g/b\rho_0)^{1/2}$ which has the dimensions of velocity. The Coriolis parameter $f = 2\Omega$ where Ω is the constant rotation rate about the z -axis, the Brunt-Väisälä frequency $N = (gb/\rho_0)^{1/2}$, $\nu = \mu/\rho_0$ is the kinematic viscosity and κ is the mass diffusivity coefficient. We assume periodic or infinite boundary conditions with Prandtl number $Pr = \nu/\kappa \sim \mathcal{O}(1)$. The relevant non-dimensional parameters for this system are the Rossby number $Ro = f_{nl}/f$ and the Froude number $Fr = f_{nl}/N$, where $f_{nl} = (\epsilon_f k_f^2)^{1/3}$ is the non-linear frequency given input rate of energy ϵ_f Smith & Waleffe (2002). Thus Ro and Fr are the ratios of rotation and stratification timescales respectively to the nonlinear timescale.

There are three overlapping ideas from turbulence theory which we will exploit in this analysis of the Boussinesq equations. First from Kolmogorov (1941) the notion of a universal range of scales governed by the dynamics of the flux of a conserved quantity. Second, the analysis of Kraichnan (1971) on the inertial ranges of two-dimensional flows based on the constraint imposed by the joint conservation of both energy and enstrophy. Kraichnan thus deduced the k^{-3} scaling, up to logarithmic correction, of the energy in

wavenumbers larger than the forcing wavenumber k_f and $k^{-5/3}$ scaling for wavenumbers smaller than k_f . Finally Charney (1971) used Kraichnan's ideas to show that similar constraints are placed on the quasi-geostrophic system due to the joint conservation of potential enstrophy and energy. Charney showed that in the quasi-geostrophic system, energy must flow upscale and potential enstrophy downscale, with predictable scaling exponents of the energy spectrum in the two ranges.

The general principle then, is that for a given (inviscidly) conserved quantity in a system there is a range of scales over which the flux of that quantity is a constant. The larger the system and the more separation there is between the source (forcing) and the sink (dissipation, boundaries), the more extended this universal 'inertial' range of scales becomes. Once this is established, it is fairly standard procedure to deduce phenomenologically what the distribution of such quantities must be in spectral space.

In the inviscid and unforced ($\mathcal{F} = \nu = \kappa = 0$), Boussinesq system the potential vorticity q is a local (Lagrangian) invariant, as is its square the potential enstrophy Q , while the total energy is a global invariant. That is,

$$\begin{aligned} \text{potential vorticity } q &= (\boldsymbol{\omega}_a \cdot \nabla \rho_T), \quad \frac{Dq}{Dt} = 0, \\ \text{potential enstrophy } Q &= \frac{1}{2}q^2, \quad \frac{DQ}{Dt} = \frac{D}{Dt} \int Q \, d\mathbf{x} = 0, \\ \text{total energy } E_T &= E + P, \quad \frac{D}{Dt} \int E_T \, d\mathbf{x} = 0. \end{aligned}$$

At any point \mathbf{x} , $E = \frac{1}{2}|\mathbf{u}|^2$ is the kinetic energy, $P = \frac{1}{2}\theta^2$ is the potential energy of the density fluctuations. The absolute vorticity $\boldsymbol{\omega}_a = \boldsymbol{\omega} + f\hat{\mathbf{z}}$ and the relative (or local) vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{u}$. PV may be written in terms of θ as

$$q = fN + \boldsymbol{\omega} \cdot \nabla \theta + f \frac{\partial \theta}{\partial z} - N\omega_3. \quad (1.4)$$

The constant part fN does not participate in the dynamics and we will therefore neglect it from now on. In what follows we will assume that $\nu \rightarrow 0$ and $\kappa \rightarrow 0$ such that

Prandtl number $Pr = \nu/\kappa = 1$, and the force \mathcal{F} is confined to the lowest modes. Thus we assume a conventional ‘inertial-range’ of turbulent scales wherein the transfer of conserved quantities dominates over both their dissipation and forcing. Note that the global potential vorticity is zero for unbounded flows. While the potential enstrophy is also Lagrangian invariant, we will use its global conservation properties in our analysis.

This set of invariants is analogous to those for 2-d turbulence where the vorticity is Lagrangian invariant and energy and enstrophy are globally conserved. It therefore seems reasonable to expect that an analysis similar to that of Kraichnan for 2D turbulence would yield similarly useful results. The raw Boussinesq system does not permit an exact relationship between energy and potential enstrophy as in the case of 2d turbulence. One notices that the potential enstrophy is, in general, a quartic quantity, while the energy is quadratic. However, when f and N are sufficiently large, PV linearizes in the dynamical variables and reduces in fourier space to:

$$\tilde{q}(\mathbf{k}) \simeq -i(fk_z\tilde{\theta} + N\mathbf{k}_h \times \tilde{\mathbf{u}}_h) = -i(fk_z\tilde{\theta} + iNk_h\tilde{u}_h) \quad (1.5)$$

where $\tilde{\cdot}$ denotes fourier coefficients, the total wavevector $\mathbf{k} = \mathbf{k}_h + k_z\hat{\mathbf{z}}$, the horizontal wavevector component has length $k_h = (k_x^2 + k_y^2)^{1/2}$, the vertical wavevector component has length k_z and \mathbf{u}_h is the horizontal velocity vector with magnitude $u_h = (u_x^2 + u_y^2)^{1/2}$. We assume that the vertical velocity $w = u_z \sim 0$ in the leading order, consistent with empirical observations, and use incompressibility to obtain the last equality of Eq. (1.5). We will next show how in the formal limits where PV is linear, asymptotically exact relationships can be extracted between potential enstrophy and components of the total energy.

First we briefly review the theoretical results that have been obtained in the limits in which PV linearizes. For strong rotation and stratification, Embid & Majda (1998) and Babin *et al.* (2000) showed that the limit as $Ro \sim Fr = \epsilon \rightarrow 0$ is QG after the inertia

gravity waves are averaged out. The scalings predicted in Charney (1971) are recovered, namely k^{-3} scaling of the energy for $k_f \ll k \ll k_d$, and $k^{-5/3}$ scaling for $k < k_f$ where k_f is the forcing wavenumber and k_d is the typical dissipation wavenumber. Embid & Majda (1998) also showed that the limiting dynamics for $Fr \rightarrow 0$ with *finite* Ro includes the so-called vertically sheared horizontal flows (VSHF). In both these cases, one can obtain a set of closed reduced equations for the linear PV dynamics from which all other quantities of interest can be derived. There is as yet no equivalent rigorous theoretical result for the unit-aspect ratio strongly rotating $Ro \rightarrow 0$ case with finite Fr , since in that limit the scalar equation 1.2, which is independent of Ro , vanishes in the leading order. This is a key reason why a (closed) reduced QG-like set of equations for this limit in the unit aspect-ratio domain has not been derived. Julien *et al.* (2006) showed that if an additional small parameter is introduced, namely the aspect-ratio of the domain, then reduced equations are possible for the vanishing Ro limit as well.

In recent theoretical work we predicted that the potential enstrophy associated with the Boussinesq rotating system exhibits universal statistical properties in the so-called ‘inertial range’ of scales (Kurien *et al.* (2006)) in physical space for the same limiting regimes of the Rossby and Froude numbers described above. Formally, keeping lowest-order terms in a perturbation expansion in powers of $\epsilon \rightarrow 0$, universal statistics are found when the potential vorticity is linear in the dynamical variables \mathbf{u} and θ . In those cases, the associated von Kármán-Howarth equation von Kármán & Howarth (1938) for the two-point correlation of q yields a simple scaling law for the third-order velocity-potential-vorticity correlation

$$\langle q(\mathbf{x})q(\mathbf{x} + \mathbf{r})(u_L(\mathbf{x}) - u_L(\mathbf{x} + \mathbf{r})) \rangle = -\frac{2}{3}\varepsilon_Q r \quad (1.6)$$

where \mathbf{r} is a physical separation scale, subscript L denotes the component along \mathbf{r} , and ε_Q is the mean dissipation rate of potential enstrophy Q . The 2/3-law (1.6) assumes that

the two-point statistics of potential vorticity have a universal small-scale statistically isotropic component for statistically steady state flows. A related result was subsequently derived for the strict QG case (beginning with the evolution equation for q_{QG}) Lindborg (2007) relating the third-order moments of potential vorticity difference and horizontal velocity difference across scale r to potential enstrophy flux, assuming axisymmetry of the small-scales, and restricted to components of the velocity that are in the horizontal plane (given that vertical velocity is strictly zero for pure QG). The numerical experiments to verify 1.6 are currently underway and will be presented in a separate work.

There have been key numerical studies of the theoretical limits described above of which we here review the most relevant to our present study. The limit of strong rotation and stratification including the case $f = N$ was examined by Bartello (1995) in a paper which studied geostrophic adjustment as a process in which wave-vortical-wave interactions acted as catalyst for highly efficient downscale transfer of energy downscale. Smith & Waleffe (2002) demonstrated the generation of slow large scales in strongly stratified turbulence with small scale forcing and the dominance of the $k_h = 0$ (VSHF) wave modes in the low wavenumbers. Waite & Bartello (2004) and Lindborg (2006) explored the effect of vortical forcing in stratified turbulence without rotation, thus highlighting the difference that 2D vs. 3D forcing can make in the evolution of the spectra. Sukhatme & L.M. (2008) showed that fairly small departures from $f/N = 1$ can have measurable effect on the scaling of the spectra for moderately low-resolution simulations. Waite & Bartello (2006*a,b*) and demonstrated the transition to QG from stratified turbulence as Ro is decreased for fixed small Fr . Remmel *et al.* (2009) proposed a novel non-perturbative model to add non-QG effects to a flow and demonstrate the importance of resolving the small scales (waves) in order to obtain even gross large-scale features such as the energy growth accurately.

We will here study rotating Boussinesq turbulence both heuristically and using data from high-resolution simulations for Ro and Fr chosen such that the potential vorticity is nearly linear, and potential enstrophy is nearly quadratic. In our approach we retain the type of physically sound arguments used by Kolmogorov, Kraichnan and Charney, namely a focus on conserved quantities (in the inviscid, non-diffusive system), existence of inertial ranges where the downscale flux of a conserved quantity (in our case potential enstrophy) governs the dynamics and consequent spectral scaling of other related quantities (energy), and the notion of a statistically universal range of small scales.

2. Spectral relationships between potential enstrophy and energy

We begin by assuming that velocity fluctuations and density fluctuations are about the same order. We can construct spectral relationships between the quadratic potential enstrophy $Q(k) = \frac{1}{2}|\tilde{q}(k)|^2$ and either the horizontal kinetic energy $E_h(k) = \frac{1}{2}|\tilde{u}_h(k)|^2$ or the potential energy $P(k) = \frac{1}{2}|\tilde{\theta}(k)|^2$ based on the parameter $\Gamma = \frac{fk_z}{Nk_h}$, which by definition resembles a local Burger number in spectra space. From Eq. 1.5, for $\Gamma \ll 1$,

$$\text{stratification dominates the PV} \quad \tilde{q}(k) \simeq iNk_h\tilde{u}_h, \quad (2.1)$$

$$\text{and potential enstrophy} \quad Q(k) \simeq \frac{1}{2}|Nk_h\tilde{u}_h|^2, \quad (2.2)$$

$$\text{giving the constraint} \quad \lim_{\kappa_h \rightarrow \infty} \int_{\kappa_h}^{\infty} Q(k)dk_h \gg N^2\kappa_h^2 \int_{\kappa_h}^{\infty} E_h(k)dk_h. \quad (2.3)$$

Eq. (2.3) is interpreted to mean that for wavevectors with large k_h , that is the ‘wide’ wavevectors, the potential enstrophy $Q(k)$ dominates the downscale (horizontal scales smaller than $1/\kappa_h$) dynamics and consequently suppresses the transfer of horizontal kinetic energy into those modes. This is analogous to the relation in 2d turbulence

$$\lim_{\kappa \rightarrow \infty} \int_{\kappa}^{\infty} \Omega(k)dk \gg \kappa^2 \int_{\kappa}^{\infty} E(k)dk \quad (2.4)$$

wherein the enstrophy $\Omega(k)$ suppresses the downscale transfer of kinetic energy $E(k)$ for sufficiently high wavenumber κ , and in turn forces the inverse cascade of energy.

In the opposite limit $\Gamma \gg 1$ one can show that,

$$\text{rotation dominates the PV} \quad \tilde{q}(k) \simeq -i f k_z \tilde{\theta}, \quad (2.5)$$

$$\text{and potential enstrophy} \quad Q(k) \simeq \frac{1}{2} |f k_z \tilde{\theta}|^2, \quad (2.6)$$

$$\text{giving the constraint} \quad \lim_{\kappa_z \rightarrow \infty} \int_{\kappa_z}^{\infty} Q(k) dk_z \gg f^2 \kappa_z^2 \int_{\kappa_z}^{\infty} P(k) dk_z \quad (2.7)$$

which is interpreted to mean that potential enstrophy suppresses the transfer of potential energy $P(k)$ into the wavevectors with large k_z , that is the ‘tall’ wavemodes. In physical space this may be loosely interpreted to mean that potential energy transfer into the small vertical scales is vertical scales.

More generally, for fixed f/N there is a transition regime when $k_h/k_z \simeq f/N$ if the wavenumber regime is large enough. Thus, in an idealized infinite wavenumber flow, for fixed f/N there will be some regime of wavevectors such that $k_h/k_z \gg f/N$ for which $\Gamma \ll 1$ and horizontal kinetic energy is suppressed in k_h according to Eq. 2.10, and a regime of wavevectors such that $k_h/k_z \ll f/N$ for which $\Gamma \gg 1$ and potential energy is suppressed according to Eq. 2.11. The transition regime is defined by a cone in wavevector space with angle θ to the vertical such that $\tan \theta = f/N$. All wavenumbers inside the cone would behave according to $\Gamma \gg 1$ and all those outside would behave according to $\Gamma \ll 1$. In our simulations in finite sized boxes, we study three extreme cases: $f/N \ll 1$ where the cone is extremely narrow (small angle) so that most wavevectors lie outside the cone giving $\Gamma \ll 1$, $f/N \gg 1$ where the cone is very broad (large angle) so that most wavevectors lie inside the cone and $\Gamma \gg 1$ and finally $f/N = 1$ where the cone-angle is 45deg to the vertical and both $\Gamma \ll 1$ and $\Gamma \gg 1$ regimes exist.

There are several noteworthy differences between the relations Eq. (2.3) and (2.7)

derived above and the 2d relation Eq. (2.4). The latter relates two (inviscidly) conserved quantities in spectral space, while the former relates the spectrum of a conserved quantity ($Q(k)$) with that of one ($E_h(k)$) that is not conserved in general. The 2d turbulence constraint forces enstrophy downscale and energy upscale. In rotating Boussinesq, for $\Gamma \ll 1$, when the horizontal kinetic energy is suppressed in the large k_h , there is no corresponding constraint in k_z for either the kinetic or potential energies; conversely for $\Gamma \gg 1$ which potential energy is suppressed in k_z , there is no corresponding constraint in k_h . Therefore there is always the possibility of a downscale transfer of energy in the unconstrained wavevector component. That is, a dual cascade of potential enstrophy and energy is possible downscale. In addition, Eq. (2.4) is isotropic in wavenumber in that it depends on the total spherical wavenumber while our new relations Eqs. (2.3) and (2.7) are highly anisotropic in wavenumber. As we will show from the data, the spectra indeed indicate that energy is transferred downscale in all cases albeit in a highly anisotropic manner.

2.1. *A scaling phenomenology*

Following Kraichnan and Charney one can naively postulate a scaling exponent for the horizontal kinetic energy and potential energy. Assuming the potential enstrophy flux governs the downscale dynamics, the energy scaling in the high wavenumber regime must depend only on the potential enstrophy dissipation rate ε_Q and the wavevector (component) magnitude. Dimensional analysis then results, for the two limits:

$$\Gamma \ll 1: E_h(k) = C_E \varepsilon_Q^{2/5} k_h^{-3} \quad (2.8)$$

$$\Gamma \gg 1: P(k) = C_P \varepsilon_Q^{2/5} k_z^{-3}. \quad (2.9)$$

where $C_{E[P]}$ is a prefactor which in principle should depend on the ratio of rotation to stratification since E_h and P are not separately conserved. In this study we focus on the

scaling in wavenumber and leave the study of the prefactor to a future work. Analysis of high-resolution data (see numerical results below and prior work in Kurien *et al.* (2008)) showed that the actual scaling exponent was much steeper than -3 , prompting a refinement of the above scaling estimates as follows.

We first suppose that part of the downscale energy transfer in both limits is not entirely suppressed. In the case $\Gamma \ll 1$ where we predict that $E_h(k)$ suppressed for large k_h , it may be that $E_h(k)$ or $P(k)$ is free to transfer into modes with large k_z . And in the case $\Gamma \gg 1$ with $P(k)$ suppressed for large k_z , it may be that $P(k)$ or $E_h(k)$ is free to fill modes with large k_h . Thus there may be a partial flux of the total energy (kinetic plus potential) to high horizontal or vertical wavenumber leading to the postulate that the spectral distribution of energy must depend on both ε_Q and energy dissipation ε_d in the small scales, where the subscript d denotes downscale energy. We will show *a posteriori* that our data supports such a hypothesis. A simple way to include the dependence on energy dissipation is to postulate that the energy scaling depends on the *ratio* $\varepsilon_Q/\varepsilon_d$ and on the wavevector component. Then dimensional analysis yields:

$$\Gamma \ll 1: E_h(k) \propto C_E \frac{\varepsilon_Q}{\varepsilon_d} k_h^{-5} \quad (2.10)$$

$$\Gamma \gg 1: P(k) \propto C_P \frac{\varepsilon_Q}{\varepsilon_d} k_z^{-5}. \quad (2.11)$$

Irrespectively of the exact value of the scaling exponent for the steep decay, for $\Gamma \ll 1$ the spectral horizontal kinetic energy distribution becomes independent of k_z , while for $\Gamma \gg 1$ the spectral potential energy distribution becomes independent of k_h . Notice also that this estimate requires that $\varepsilon_Q/\varepsilon$ remains finite in the inviscid $\nu \rightarrow 0$, non-diffusive $\kappa \rightarrow 0$ limit. One could presumably add corrections to mollify the blow-up for small k_h and k_z respectively along the lines of the log-correction derived by Kraichnan (1971). This issue of scaling correction will not be addressed in this paper except to say that steeper

than -3 scaling implies even stronger non-local transfers than 2d turbulence as discussed by Kraichnan (1971)). As we will show in the sections below, our modified assumptions about energy transfer downscale are supported by the results of our simulations, and the scaling exponents measured from our data for the appropriate regimes are remarkably close to -5 .

Bellet *et al.* (2004) studies mode-angle dependence of the total energy spectrum of rapidly rotating turbulence flow, using closure models for the anisotropic rotating equations for wave turbulence. The flows they study do not have stratification or any advected scalar and thus have no potential vorticity. Therefore, our present analysis, which does in effect study spectra as function of mode-angle via k_z/k_h , cannot be carried over to deduce similar constraints. What we will present poses several departures from Bellet *et al.* (2004). First, we address, in principle, a very broad parameter space of finite Ro and Fr at high Reynolds numbers via the parameter Γ . We also, for the first time, show that potential enstrophy conservation imposes constraints on the energy distribution in parameter regimes other than strict QG. And finally we propose a way to understand how a conserved quantity (potential enstrophy) may affect the spectral distribution of a non-conserved quantity (horizontal kinetic energy or potential energy) and furthermore propose a consistent argument to estimate the resulting spectral scaling exponents.

3. Numerical simulations and results

We perform pseudo-spectral calculations of the Boussinesq equations with rotation on grids of 640^3 points and 1024^3 points in unit aspect-ratio domains. The computational domain has length $L = 1$ to a side, with wavenumbers in integer multiples of 2π . The time-stepping is 4th-order Runge-Kutta with resolution of the smallest wave frequencies with five timesteps per wave period. In the simulations at 640^3 , the diffusion of both

momentum and density (scalar) is modeled by hyperviscosity of laplacian to the 8th-power in order to extend the inertial scaling ranges. The Reynolds number in these cases is not computed explicitly but the hyperviscosity coefficient is chosen to resolve the total energy in the largest shell Chasnov (1994); Smith & Waleffe (2002) thus ensuring adequately turbulent flow for a given resolution. The simulations at 1024^3 were run with regular Navier-Stokes viscosity. The energy input rate $\epsilon_f = 0.5$ and 1 respectively for the smaller and larger resolutions, and Ro and Fr are varied by varying the rotation and stratification rates for fixed energy input rate which determines the nonlinear timescale. The forcing is incompressible and equipartitioned between the three velocity components and θ . This is equivalent to forcing the two wavemodes and the one vortical modes equally, thus providing an optimally unbiased calculation. The forcing is also stochastic and the phases of each mode change sufficiently rapidly that the forced modes are statistically isotropic. The forcing spectrum is peaked at $k_f = 4 \pm 1$, for large scale forcing. The simulations are dealiased according to the isotropic two-thirds dealiasing rule. Lower resolution runs at 256^3 and 512^3 (see Kurien *et al.* (2008)) were also performed in earlier studies, but are not reported here.

We seek to verify our predictions for three particular “extreme” regimes in Ro , Fr (see Table 1). The first is strongly stratified with moderate rotation, $f/N \ll 1$ (runs 1 and 4); the second regime has strongly rotating with moderate stratification, $f/N \gg 1$ (runs 2 and 5); and finally the regime which fixes $f = N$ large, (run 3) so that the dependence of potential vorticity on the ratio Γ is reduced to a dependence on the cotangent $\tau = k_z/k_h$ alone, of the angle of the wavevector with respect to the vertical. Presumably, for sufficiently long inertial ranges any combination of f , N , k_h and k_z such that $\Gamma \gg$ or $\ll 1$, while maintaining near-linear PV, may be similarly investigated. As discussed in Bartello (1995), strict potential enstrophy conservation is not possible

in a truncated inviscid spectral scheme due to the lack of conservation of Lagrangian invariants by individual triads. The best we can expect with this method in the viscous case is that truncation errors are swamped by viscosity thus giving conserved fluxes in the inertial range. We mention this fact because we will be using the notion of potential enstrophy conservation in this study. The spectral method itself is routinely used to compute flows very similar to those we study here and we proceed while remaining aware of the caveats.

For each case in the Table, we will present the calculations to check specific predictions of the phenomenology proposed, namely the anisotropic suppression of horizontal kinetic energy and potential energy in the different Γ regimes, and the scaling exponents predicted in these regimes by Eqs. (2.10) and (2.11). The spectra as a function of (k_h, k_z) are defined as follows:

$$\frac{1}{2}|a(k)|^2 = \frac{1}{2}|a(k_h, k_z)|^2 = \frac{1}{2} \sum_{k_h - 0.5 \leq \kappa_h < k_h + 0.5} |a(\kappa_h, k_z)|^2 \quad (3.1)$$

where a is any of \tilde{q} , \tilde{u}_h or $\tilde{\theta}$. We will present the horizontal kinetic energy and potential energy spectra (a) as a function of k_h for various values of fixed k_z , (b) as a function of k_h , summed over k_z , (c) as a function of k_z for various values of fixed k_h , and (d) as a function of k_z , summed over k_h . The (a)-(d) above correspond to subplots (a)-(d) in the sets of figures to follow. In each subplot, horizontal kinetic energy is plotted on top and potential energy on the bottom. We will also show the more conventional decomposition of the energy in contributions from the linear eigenmodes of the Boussinesq system. There are two inertial-gravity wave-eigenmodes with non-zero frequency and zero PV which contribute to the wave (or ageostrophic) energy spectrum. The remaining eigenmode has zero frequency and linear PV and contributes to the vortical (or geostrophic) energy spectrum. We will attempt to relate the mode-angle representation to the usual wave-vortical decomposition. Since this paper is largely an exploration of spectral scaling in

run	grid	diffusion	ϵ_f	k_f	f (Ro)	N (Fr)
1	640^3	hyper	0.5	4	14 (0.48)	3000 (0.0023)
2	640^3	hyper	0.5	4	3000 (0.0023)	14 (0.048)
3	640^3	hyper	0.5	4	3000 (0.0023)	3000 (0.0023)
4	1024^3	NS	1	4	8.58 (1)	8580 (0.001)
5	1024^3	NS	1	4	4290 (0.002)	8.58 (1)

TABLE 1. The parameters for the numerical simulations of the Boussinesq equations. The notation “hyper” denotes hyperviscous diffusion of Laplacian to the eighth power, applied to both momentum and scalar; “NS” denotes regular Navier-Stokes viscosity.

the large wavenumbers, we will, for clarity’s sake, use the hyperviscous simulations data with their extended ranges to compute scaling exponents. The Navier-Stokes cases (runs 4,5) show no appreciable differences from their hyperviscous counterparts except for shortened scaling ranges, as is to be expected. Wherever appropriate, and when it does not result in a proliferation of figures, we will also present the data from the simulations with Navier-Stokes viscosity.

First we show in Fig. 1 that in each of the three cases, the global potential enstrophy (solid line) is captured by one half of the square of the corresponding linear PV, to within 3% or better agreement. potential enstrophy, after a period of spin-up and over-shoot, settles to a constant, indicating statistically steady state for this quantity, the mean rate of input of potential enstrophy balancing the mean rate of dissipation. We are thus able to establish the time-frame over which the small scales, governed by potential enstrophy dynamics, have achieved statistically steady state. The small scale spectra of energy that we are interested in, achieve a statistically steady state. The total energy meanwhile (not shown) continues to grow since it can transfer upscale, where in our case, there is no sink to drain the energy. The calculation of all statistical quantities in what follows occurs

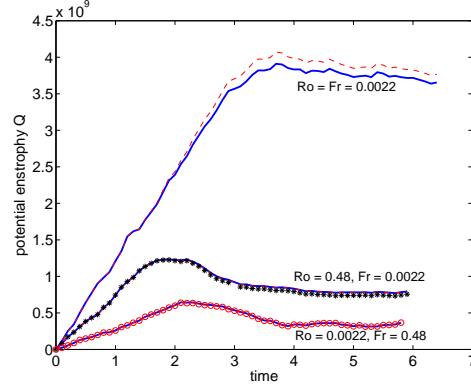
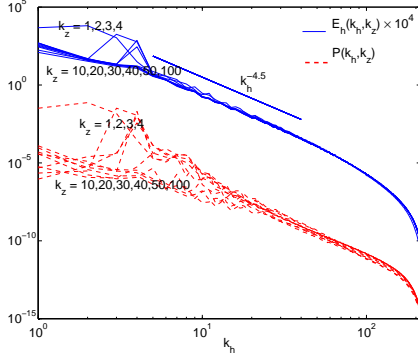


FIGURE 1. [Color online] Global potential enstrophy evolution in time for runs 1,2, and 3. The total Q and its quadratic components corresponding to linear q , are shown. Solid line: total Q ; dashed line : $\frac{1}{2}|f\frac{\partial\theta}{\partial z} + N\omega_3|^2$; stars: $\frac{1}{2}|N\omega_3|^2$; circles: $\frac{1}{2}|f\frac{\partial\theta}{\partial z}|^2$. In each limiting case, the corresponding quadratic part of the potential enstrophy captures the total Q to within 3%.

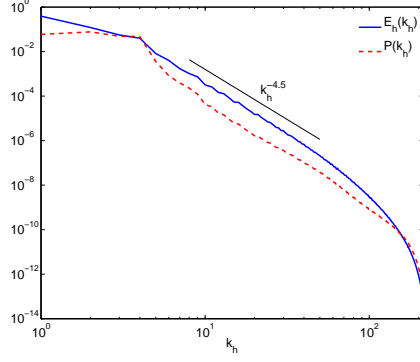
over the time-frame where the mean potential enstrophy is constant, $3.5 < t < 6$, over about 25 frames.

3.1. Strong stratification, moderate rotation

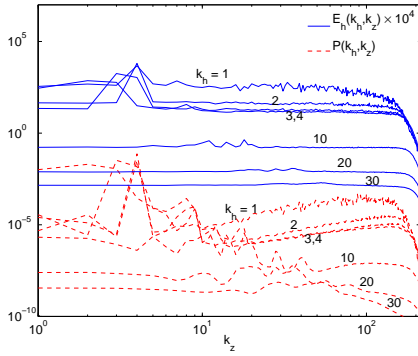
This case is studied using data from runs 1 and 4 of Table 1. We are in the limit $\Gamma \ll 1$ since nearly all wavevectors lie outside the narrow cone with vertical angle $\tan\theta = f/N = 0.0047$. Therefore, $\tilde{q}(k) = \tilde{q}(k_h, k_z) \sim Nk_h\tilde{u}_h$ irrespective of k_z/k_h and the spectral scaling of $E_h(k)$ should become independent of k_z as given by Eq. 2.10. Fig. (2(a)) (top) and shows that $E_h(k_h, k_z)$ collapses to a function of k_h alone over the wide range $0 < k_z < 100$, and scales very close to k_h^{-5} . In fact, the collapse is so complete that the



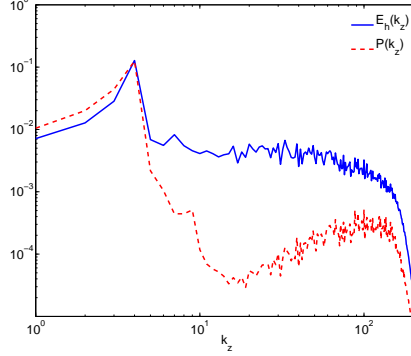
(a) Spectra of horizontal kinetic energy (solid lines) and potential energy (dashed lines) as a function of k_h for various fixed $1 \leq k_h \leq 100$ as indicated.



(b) Spectra of horizontal kinetic energy (solid line) and potential energy (dashed line), as functions of k_h , after summing over k_z .



(c) Spectra of horizontal kinetic energy (solid lines) and potential energy (dashed lines) as a function of k_z for various fixed $1 \leq k_h \leq 30$ as indicated.



(d) Spectral of horizontal kinetic energy (solid line) and potential energy (dashed line), as functions of k_z , after summing over k_h .

FIGURE 2. [Color online] The horizontal kinetic energy and potential energy spectra for $f \ll N$ moderately rotating and strongly stratified flow (Table 1 run 1).

converse plotting of E_h as a function of k_z in Fig. 2(c)) for various fixed k_h , gives a series of plateaus, indicating no k_z dependence at all in the inertial range for any particular k_h . The corresponding plots for $P(k_h, k_z)$ shows that it nominally follows the scaling of E_h as a function of k_h (Figs. 2(a)), yielding k_z -averaged scaling of k_h^{-5} (Fig. 2(b)). However the converse plotting of P (Fig. 2(c)) shows that it does not become completely independent of k_z even though, when *summed* over k_z (Fig. 2(d)), its scaling becomes close to k_h^{-5} as predicted in this limit for $E_h(k_h)$. Thus P appears to mimic the scaling of E_h as a function of k_h , but not as a function of k_z .

We now make a more conventional representation of the spectra by plotting the energy contained separately in the wave and vortical modes as a function of the spherical wavenumber k . Figure 3 shows that almost the entire downscale energy is captured by the vortical modes. The narrow upscale range is also filled by the vortical modes. The total energy (vortical energy) distribution is nearly a constant as a function of k . The wave energy is subdominant everywhere except in the forced modes, indicating that the wave energy does not move efficiently out of those modes in this strongly stratified regime. This figure is to be compared with Fig. 2(d) to see that for $k > k_f$ the vortical modes mimic the horizontal kinetic energy and that the independence of the latter from k_z translates to a constant spectral distribution over k as well, given that the dependence of E_h on k_h is rapidly very decaying (Fig. 2(b)). To the best of our knowledge, this is the first observation and explanation of the asymptotic high wavenumber energy spectrum for strongly stratified flow with finite rotation, with isotropic forcing in the low-wavenumbers (large scales). Waite & Bartello (2004) observed k_h^{-5} scaling for the total energy spectrum and the vortical mode energy spectrum in flows with rather low stratification N , zero rotation and forced in the vortical modes only. They also observed saturation in k_z of the total energy spectrum. In the light of the analysis in the present work, it would appear they

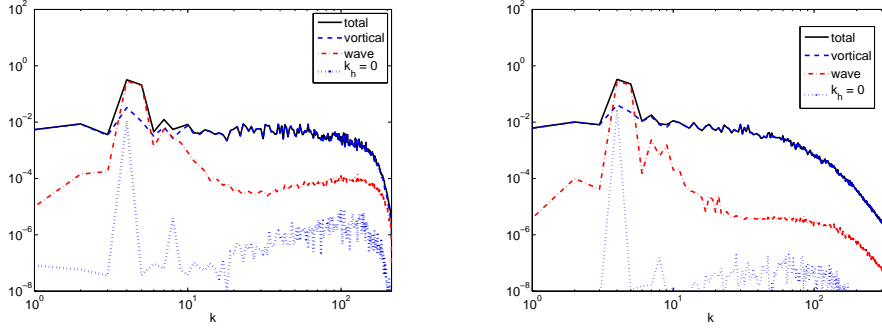


FIGURE 3. [Color online] Strongly stratified and moderately rotating cases, with hyperviscosity (run 1, left) and with Navier-Stokes viscosity (run 4, right). Total (black solid), wave (red dash), vortical (blue dot-dash) and $k_h = 0$ mode (blue dotted) energy spectra. Almost the entire total energy is captured by the vortical modes within which the energy is uniformly distributed as k^0 . The wave-mode contribution dominates in the forced modes only. The $k_h = 0$ modes form a negligible contribution, highlighting the difference from VSHF.

achieve something like the regime $\Gamma \ll 1$ and linear $q \simeq N\omega_3$, not with large N , but with large ω_3 achieved by forcing vortical modes only. Their paper Waite & Bartello (2006b) which studies similar parameter regimes in Fr while varying Ro does show k_z^0 scaling of the vortical mode energy for $Ro \sim \mathcal{O}(1)$ but does not recover the k_h^{-5} steep decay. As we have pointed out above, the exact decay exponent does not matter as long as it is a decay and is independent of k_z , leading to the k_z^0 scaling. Our analysis goes a step further in proposing an explanation of *why* there is suppression in k_h and not in k_z due to the very particular constraints imposed on energy by the potential enstrophy, leading to independence from k_z . We have also conclusively shown that the asymptotic small Fr

regime for the small scales can be achieved with unbiased isotropic forcing and in the presence of finite rotation.

There is also a stark difference from the vertically sheared horizontal flows (VSHF) ($k_h = 0$ modes) expected to be included in this limit when the forcing is at *high* wavenumbers (small scales). In our data the $k_h = 0$ modes contain a very small percentage of the total energy (see Fig. 3) and the vortical (PV) modes contain most of the energy, whereas the opposite is true for the low wavenumbers in the case where the forcing is in the high wavenumbers (Smith & Waleffe (2002); Embid & Majda (1998)).

3.2. Strong rotation, moderate stratification, $Ro = 0.0022$, $Fr = 0.48$

For run 2 of our simulations with moderate stratification and strong rotation, $\Gamma \gg 1$ and we expect to observe Eq. (2.11). In Fig. 4(a) we see that although the scaling of $P(k)$ in k_z for all k_h shows some degree of agreement, the curves themselves have not collapsed. This is also shown by the converse plotting of $P(k)$ for various k_z functions of k_h in Fig. 4(a). However we note that the trend as Ro decreases is for the potential energy curves to collapse to functions of k_z alone. Thus $Ro = 0.002$ may not be sufficiently small to see the limiting small-scale dynamics for $Ro \rightarrow 0$, $Fr \sim \mathcal{O}(1)$. And this despite the fact that potential enstrophy has converged to the stratification dependent quadratic component as seen in Fig. 1. Thus it appears, at least empirically, that for the small scales, the asymptotic regime of strong stratification is easier to achieve than that of strong rotation. This would seem related to the fact that the $Ro \rightarrow 0$ limit for finite Fr is not amenable to a reduced set of equations since in that limit the scalar evolution equation vanishes in the leading order.

Figures 4(b) and 4(d) show that the k_z -summed spectra scale between k_h^{-2} and $k_h^{-4/3}$ and the k_h -summed quantities scale approximately as k_z^{-5} . Thus, on average both horizontal kinetic and potential energies are suppressed in the large k_z modes while both are

free to fill the large k_h modes. Again this justifies the ansatz that the scaling depends on $\varepsilon_Q/\varepsilon_d$, with downscale fluxes of both potential enstrophy and energy fluxes playing a role in the final development of the high wavenumber spectra.

The conventional representation of wave and vortical mode spectra, Fig. 5 shows that in this regime, the vortical modes once again dominate the contribution to the total energy at all wavenumbers except around the forced wavenumbers, where the wave-mode contribution dominates. At high wavenumbers, the vortical modes scale more or less as $k^{-5/3}$. When compared with Fig. 4(d) this may be interpreted as the vortical modes being dominated by the potential energy, which persists strongly into the large k_h modes while decaying rapidly in the large k_z modes. Thus in the aggregate the k_h scaling dominates in the spherically averaged spectra as well. On the right of Fig. 5 shows the same decomposition for the run 5 in the strongly rotating regime which shows qualitative agreement with the hyperviscous case, comparable scalings and hence providing some confidence that in these cases hyperviscosity does not bias our interpretation with undesirable artifacts.

The point of similarity between the strongly stratified versus the strongly rotating limits is that in both cases the vortical modes dominate the high wavenumbers or small scales. However, in the former the vortical modes carry the horizontal kinetic energy whereas in the latter the vortical modes carry the potential energy. This is very different from QG wherein the vortical mode energy decays as k^{-3} in the high wavenumbers. Note also the difference from the equally strongly rotating and stratified case (see below) which, as is well known (see Bartello (1995)), carries most of the high wavenumber energy in the wave modes which scale as k^{-1} .

3.3. Equally strong rotation and stratification, $Ro = Fr \sim 0.0022$

In the special case of $f = N$ the dependence of PV on Γ reduces to a dependence on the cotangent $\tau = k_z/k_h$ of the wavevector angle. As $\tau \ll 1$, the N -dependent part prevails

while for $\tau \gg 1$, the f -dependent part takes over. That is, the cone in k -space separating $\Gamma \ll 1$ and $\Gamma \gg 1$ makes a 45° angle to the vertical allowing equal numbers of modes in both regimes. Using run 3, we show that as $\tau \ll 1$ the value of $|\tilde{q}(k)| \simeq N|k_h \tilde{u}_h(k)|$ (Fig. 6, left panel), while as $\tau \gg 1$ the value of $|\tilde{q}(k)| \simeq f|k_z \tilde{\theta}(k)|$ (Fig. 6, right panel). Since there is thus an average over the annulus of horizontal wavenumbers contained in $k_h \pm 0.5$ the curves on the left panel of Fig. 6 are smoother than those on the right. The special case of $k_h = 0$ has been omitted for clarity on the right panel as it is noisy (less averaging), while the $k_z = 0$ case on the left poses no such problems. This figure clearly shows the dependence of $\tilde{q}(k) = \tilde{q}(k_h, k_z)$ on the mode-angle with respect to the vertical as expected by Eq. (1.5) for $f = N$ large.

Next we consider the effect on the scaling of energy spectra as predicted in Eqs. (2.10, 2.11). Fig. 7(a) shows E_h and P as a function of k_h for various fixed k_z . For E_h , the spectrum for a particular k_z changes over from basically flat to slowly increasing for small k_h to close to k_h^{-5} decay when $k_h > k_z$. This is in agreement with the deduction that horizontal kinetic energy is suppressed in the small aspect ratio modes by the dominance of potential enstrophy in those modes, and consequently scales according to Eq.(2.10). Note that there is a cross-over range below which the energy decreases as k_z increases, and above which the energy increases as k_z increases. In Fig. 7(a) the spectra for potential energy for which we do not have a prediction in this limit of aspect ratio, when plotted in the same manner as above, also shows aspect-ratio dependence for $\tau \ll 1$ but with much shallower scaling, the k_z^{-2} line is shown for comparison. There is no cross-over range for $P(k_h, k_z)$ and there appears to be much less suppression of potential energy for modes such that $\tau \ll 1$.

In Fig. 7(b) we show $E_h(k_h)$ summed over all k_z (top) and similarly $P(k_h)$ (bottom). Interestingly, these scale approximately as $k_h^{-4/3}$ and k_h^{-1} respectively, which is much

shallower than the k_h^{-5} observed or E_h for $\tau \ll 1$, and even shallower than $k_h^{-5/3}$. Thus, while for a given k_z the horizontal energy is suppressed for $\tau \ll 1$, the *average* over k_z indicates net transfer to large k_h of both E_h and P .

To verify the prediction Eq. (2.11) for the large aspect-ratio modes $\tau \gg 1$ we examine the scaling of $P(k)$ as a function of k_z for various fixed k_h (Fig. 7(c)). Here again the spectrum is nearly a plateau with a turnover to k_z^{-5} as $\tau \gg 1$. Note again the cross-over range below which potential energy decreases as k_h increases, and above which the potential energy increases as k_h increases. $E_h(k)$ plotted in the same way also shows some aspect ratio dependence but with much shallower scaling indicating that E_h is not suppressed in the wavevectors with $\tau \gg 1$.

In Kurien *et al.* (2008) we presented the spectral scaling results for the case $f = N$ for runs at 512^3 and all other parameters very similar, except that in that case only the momentum was forced. In that paper, we did not have an estimate for steeper than -3 scaling exponent. The resulting spectra behaved very similarly to what we have presented in Fig. 7 so far as scaling exponents are considered. The overall magnitude of the potential energy spectra are much higher in the scalar-forced case presented here, as is to be expected. Thus the detailed forcing seems not to affect the results as far as the scaling exponents go.

To conclude this discussion for the $f = N$ strongly rotating and stratified case, the suppression of E_h in the small aspect ratio modes \mathbf{k} with $\tau \ll 1$ does not result in potential energy being suppressed in the same way. Conversely, the suppression of P in the large aspect ratio modes with $\tau \gg 1$, does not result in the horizontal kinetic energy being suppressed in those modes. However, on *summing* over all k_z [k_h] because of the strong aspect-ratio dependence, $E_h(k_h)$ [$P(k_z)$] scales with exponent between -1 and $-4/3$. Thus a mode-by-mode suppression of horizontal energy in the small aspect-ratio

modes and of the potential energy in the large aspect-ratio modes, nevertheless results in a net transfer of these energies into the high wavenumbers *on average*, along with the downscale transfer of potential enstrophy. We thus obtain *a posteriori* justification in our ansatz of $\varepsilon_Q/\varepsilon_d$ dependence (Eqs. (2.10) and (2.11)) for the functional form of E_h and P .

This example also goes to show how one-dimensional spectra in either the k_h or k_z wave-component may obscure underlying dependencies on aspect-ratio of the modes and hide the subtle ways by which the energy populates the high wavenumbers on average. This possibility was noted in Bellet *et al.* (2004) who studied purely rotating flow, with no PV, and noticed a dependence of the spectra on the mode-angle as well.

4. Flow visualizations

To give a qualitative picture of the quantities of interest, we present volume rendering of snapshots of the fully developed flow for the horizontal kinetic energy and the potential energy in Fig. 4. It must be noted that it is not clear what correspondence the spectra have to the structures in the physical space visualizations of the flow. The spectra are related to the physical space correlations as fourier transform pairs. A particular distribution of a field in physical space does not indicate a unique spectral distribution in fourier. The visualizations presented here should be treated as qualitative impressions of the flow structures. Nevertheless, we will point out consistencies between what is observed in the spectra with what is observed in visualizing the corresponding physical space quantities.

When stratification dominates over rotation the horizontal kinetic energy exhibits flat horizontal structures at all values (Fig.9(a)) which appear to have very little extent in the vertical direction. This is consistent with both Eq. 2.10 and the discussion and plots in section 3.1 which confirm complete independence from k_z . These structure are quite

different from VSHF since the $k_h = 0$ modes have a very small contribution in this regime. The potential energy for this flow (Fig. 9(b)) appears to exhibit some tendency towards horizontally oriented blobs but retains more three-dimensionally coherent structures at the larger values, indicating that the potential energy distribution follows the distribution of horizontal kinetic energy to a degree, consistent with Fig. 2(b) but still retains some dependence on k_z for small k_z (large vertical scales) consistent with Fig. (2(d)).

Conversely, when rotation dominates over stratification, Fig. 9(d) shows that the potential energy exhibits structures that are strongly oriented in the vertical direction at all values with very little extent in the horizontal, consistent with Eq. (2.11). The horizontal kinetic energy also shows vertically coherent structures Fig. 9(c) but also shows high-valued larger blobs with more horizontal extent showing that E_h also retains some horizontal scale dependence.

In the case of equal rotation and stratification (Figs. 9(e), 9(f)), horizontal kinetic energy and potential energy look rather isotropically distributed with no preferred direction, in agreement with the spectra in Figs. 7(b) and 7(d) which show relatively shallow (ranging between -1 and -4/3) scaling in both k_z and k_h on average.

To summarize, equal rotation and stratification does not show any preferential direction for the energy isosurfaces, consistent with nearly the same shallow scaling in k_h and k_z (Figs. 7(b), 7(d)). Stratification dominated flows align the energy isosurfaces in the horizontal direction while rotation dominated flow align the energy isosurfaces along the vertical. The strongly stratified data are more in agreement with our asymptotic predictions than the strongly rotating data; the latter observation could indicate that even stronger rotation is required to see the asymptotic behavior.

5. Discussion and Conclusions

The small scale, high wavenumber spectra of rotating Boussinesq have never before been studied at such asymptotically low values of Ro and Fr using high-resolution simulations with unbiased isotropic forcing and dissipation. The rigorous theoretical results for the $Ro \sim \mathcal{O}(1)$, $Fr \rightarrow 0$ by Embid & Majda (1998) and the corresponding simulations of Smith & Waleffe (2002) extract the $k_h = 0$ wave-modes as the VSHF contribution to the large scales. In our study of the small scales in this limit, which we have checked have negligible contribution of the $k_h = 0$ modes (see Fig. 3), the vortical modes display a constant spectrum for $k > k_f$, apparently dominated by the k_z^0 behavior of the horizontal kinetic energy (see Fig. 2(d)). This appears to be a consequence of one of the key new results of this work, namely that the horizontal kinetic energy becomes independent of (constant in) k_z for any k_h (see Eq. 2.10 and Fig. 2(c)) for $\Gamma \ll 1$. This uniform spectral distribution implies a very efficient downscale transfer, with energy contained predominantly by the vortical modes. The exact value of the steep scaling exponent in k_h is not as important for this result as the fact that the spectrum becomes independent of (constant in) k_z while decaying sufficiently quickly in k_h such that the spectrum becomes independent of spherical wavenumber k as well (Fig. 3).

The strongly rotating case with finite stratification (runs 2 and 5) were expected to achieve the limiting behavior of Eq. 2.11 with collapse of the spectral curves for all k_h as a function of k_z . While we see the correct trend in this direction, the curves in Fig. 4(a) show that the collapse is not complete. We are thus lead to believe that Ro will need to be driven even lower and the resolution increased to see the collapse. We can postulate that, for sufficiently small Ro , the potential energy curves for various k_h will collapse to the same function of k_z , corresponding to Eq. 2.11. Then, potential energy as a function

of k would be flat, governed by the flat spectrum in k_h and strongly decaying spectrum in k_z . Again the exact exponent of the decay law would not affect this general conclusion.

The equally strongly rotating and stratified case (run 3) show how underlying anisotropic scalings of the spectra, depending on the cotangent of the mode-angle τ , can be masked by summing over one or other of the wavevector components, as is commonly done. In Fig. 7(b) we show that the sum over all k_z of the horizontal kinetic energy $E_h(k_h)$ shallow with scaling $\sim k_h^{-4/3}$. Thus the *average* over all k_z smears out the effect of suppression by potential enstrophy and results in a much shallower scaling than the τ dependent k_h^{-5} scaling. The net effect is that energy fills the high-wavenumbers with almost the same shallow scaling in k_h and k_z (Fig. 7(b) and 7(d)). In Fig. 8 the wave-vortical mode decomposition of the spectra shows that the total downscale energy scales as k^{-1} and is dominated by wave modes. This is a well-known result and the scaling is identical to that of the passive scalar since the wave-modes are passively advected in this limit. What we further establish from this analysis is that the shallow spectrum is set up in spite of the underlying steeper, anisotropic, τ -dependent spectra in k_h and k_z .

We have made an attempt at a physically consistent phenomenology that results in a steeper than -3 scaling of the spectra within this framework. The -5 scaling that arises out of the assumption that the ratio $\varepsilon_Q/\varepsilon_d$, where ε_d is the net flux of the energy downscale, plays a role in the downscale dynamics seems justified *a posteriori*. The analysis and verification of such an ansatz requires a detailed study of the fluxes and dissipation of both potential enstrophy and energy and is the subject of a separate work. As we have said before, the main conclusions of this paper hold irrespective of the specific scaling exponent for the decay of energy in the k_h and k_z .

The prediction for steep decay in k_h according to Eq. (2.10) arises from the constraint that potential enstrophy places on the horizontal kinetic energy in k_h for $\Gamma \ll 1$. While

there is no explicit constraint on the potential energy, the data analysis in this regime reveals that potential energy scales very similarly to the horizontal kinetic energy as a function of k_h , indicating that it too is suppressed in k_h . Thus the total energy, where the contribution from the vertical component of the velocity is measured to be very small, appears to be suppressed in the large k_h modes. The observed shallow scaling in k_z of both horizontal kinetic and potential energies, suggests that the total energy is allowed to transfer into the large k_z modes. Similarly, for $\Gamma \gg 1$, the suppression of potential energy in k_z is mirrored to a large degree by the horizontal kinetic energy. These observations indicate that the exchange between kinetic and potential energies is local in wavenumber in both $f/N \ll 1$ and $f/N \gg 1$ regimes.

Recent work explored the connection between hyperviscosity, Galerkin truncation and the energy bottleneck Frisch *et al.* (2008) in Navier-Stokes turbulence without rotation or stratification. In that paper it was proposed that high values of hyperviscosity result in a partial thermalization of the high wavenumbers which manifests as energy ‘bottleneck’ phenomenon which contaminates the inertial range. In the present studies of rotating and stratified turbulence, the only substantial difference between the hyperviscous and the comparable runs with regular NS viscosity is the consistently less extended scaling ranges of the latter runs, which is to be expected. Therefore we use the hyperviscous runs to extract scaling exponents while checking against the NS runs to make sure that bottleneck-type artifacts are not introduced. It appears that for rotating and stratified flow and for the value of hyperviscous power used, the bottleneck-type artifacts, if any are minimal, and do not affect the purposes of this study.

While we have discussed three extreme parameter regimes, this discussion encompasses all values of Γ thus allowing us to analyze arbitrary f/N in the limit of linear (or near linear) PV. In an idealized infinite domain where $k \rightarrow \infty$, for any fixed f/N the $\Gamma \ll 1$

regime is achieved when $k_h/k_z \gg f/N$ while the $\Gamma \gg 1$ regime is achieved when $k_h/k_z \ll f/N$. The transition between regimes occurs for $\Gamma = 1$ or when $1/\tau = k_h/k_z = f/N$. Thus we observe that this type of analysis could be very relevant to geophysical flow regimes where f and N values are not as extreme but where the scaling ranges are much longer.

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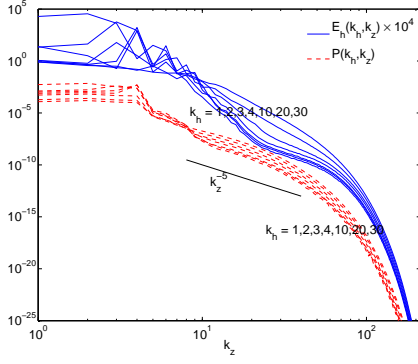
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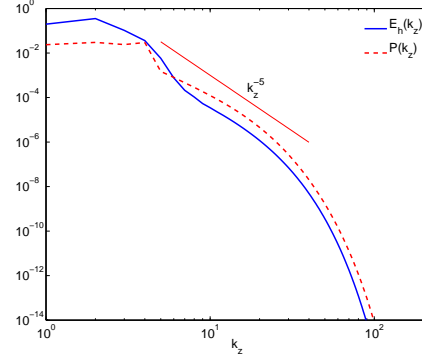
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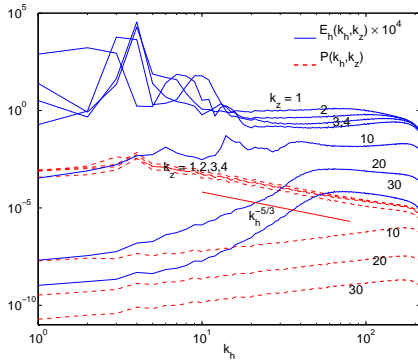
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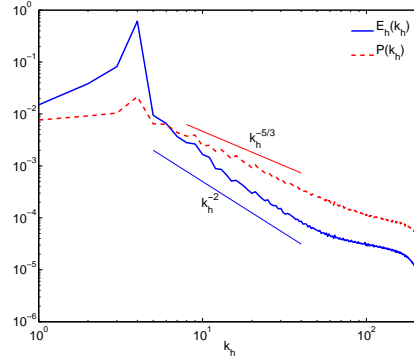
(a)



(b)



(c)



(d)

FIGURE 4. [Color online] The horizontal kinetic and potential energy spectra for $f \gg N$ strongly rotating and moderately stratified flow (Table 1 (ii)). The quantities plotted for subfigures (a,b,c,d) are the same as those for Fig. 2(c,d,a,b) respectively.

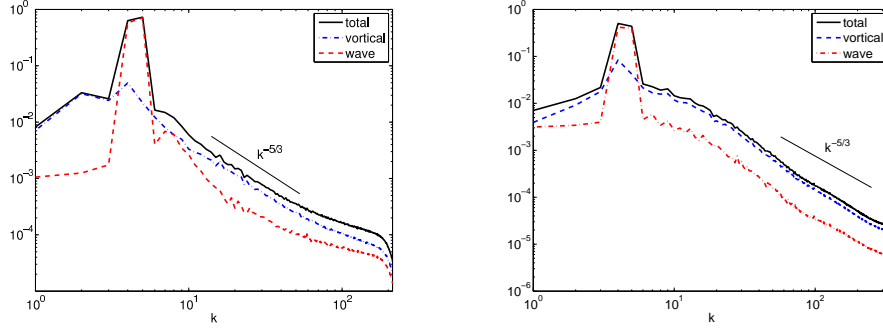


FIGURE 5. [Color online] Strongly rotating and moderately stratified cases, with hyperviscosity (run 2, left) and with Navier-Stokes viscosity (run 5, right). Total (black solid), wave (red dash), vortical (blue dot-dash) energy spectra. Almost the entire total energy is captured by the vortical modes within which the energy is uniformly distributed approximately as $k^{-5/3}$. The wave-mode contribution dominates in the forced modes only.

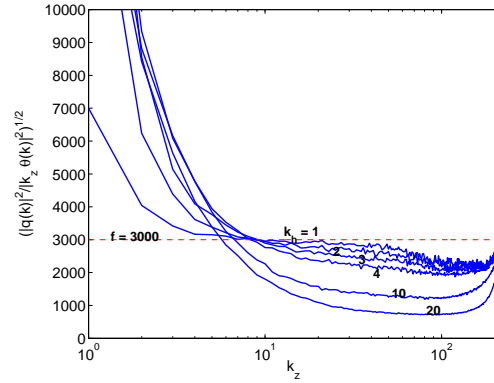
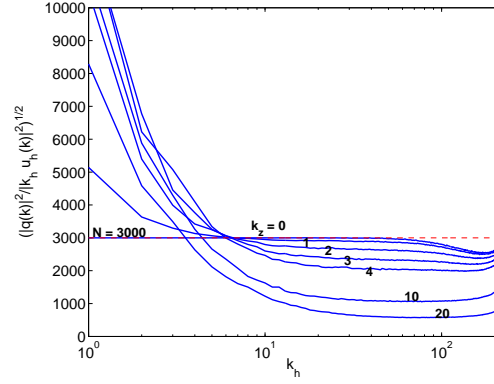
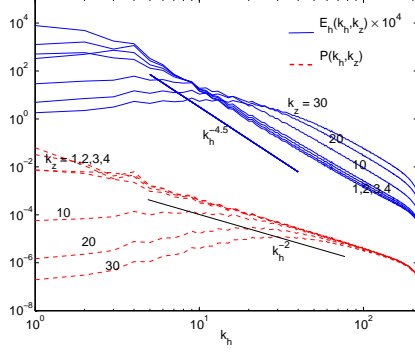
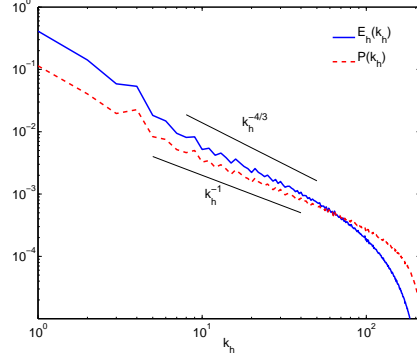


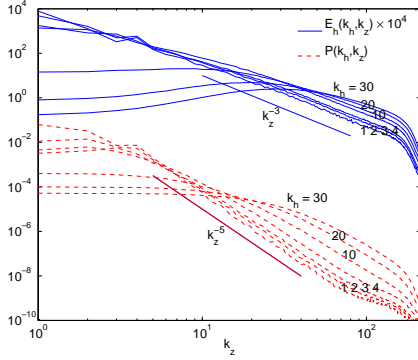
FIGURE 6. [Color online] Run 3, $f = N = 3000$ ($Ro = Fr = 0.0022$). Left: $|\tilde{q}(k)|$ normalized by $|k_h \tilde{u}_h(k)|$ as a function of k_h for various values of k_z . The normalized potential vorticity asymptotes to $N = 3000$ for $\Gamma = \tau \ll 1$ as predicted in Eq. (2.1). Right: $|\tilde{q}(k)|$ normalized by $|k_z \tilde{\theta}(k)|$ as a function of k_z for various values of k_h . The normalized potential enstrophy asymptotes to $f = 3000$ as $\Gamma = \tau \gg 1$ as predicted in Eq. (2.5)



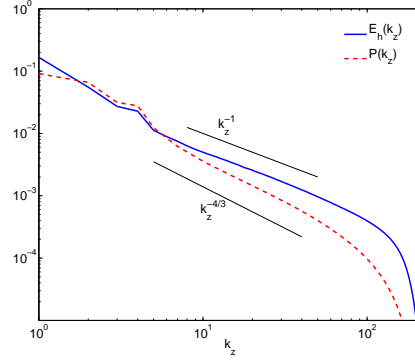
(a) Spectra of horizontal kinetic energy (solid lines) and potential energy (dashed lines) as a function of k_h for various fixed $1 \leq k_z \leq 30$ as indicated.



(b) Spectra of horizontal kinetic energy (solid line) and potential energy (dashed line), as functions of k_h , each summed over k_z .



(c) Spectra of horizontal kinetic energy (solid lines) and potential energy (dashed lines) as a function of k_z for various fixed $1 \leq k_h \leq 30$ as indicated.



(d) Spectra of horizontal kinetic energy (solid line) and potential energy $P(k_z)$ (dashed line), each summed over k_h , as functions of k_z .

FIGURE 7. [Color online] The horizontal kinetic and potential energy spectra for $f = N$ strongly rotating and stratified flow (Table 1 data (i)).

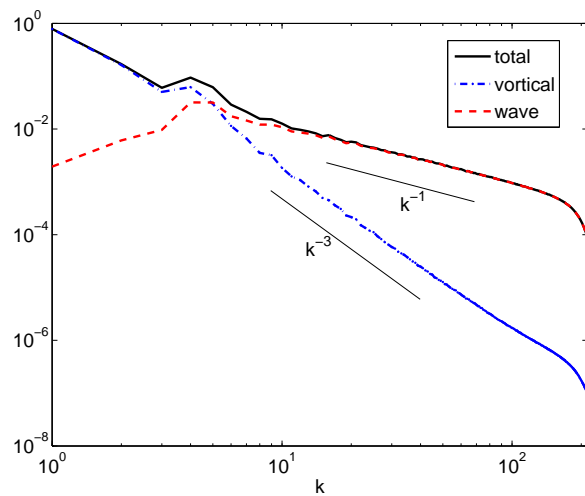
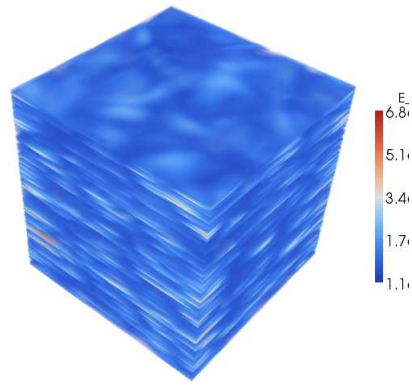
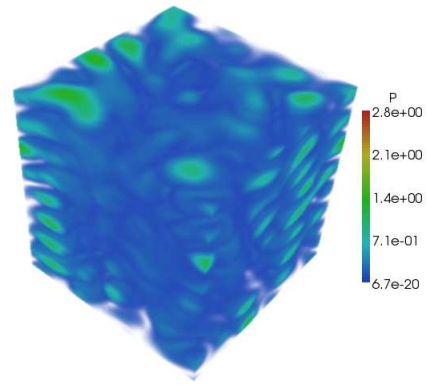


FIGURE 8. [Color online] Wave, vortical and total spectra for run 3 $Ro = Fr = 0.0023$.

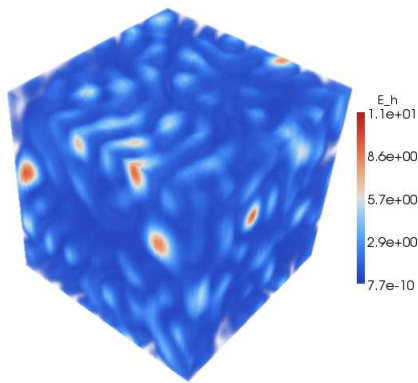
[ht]



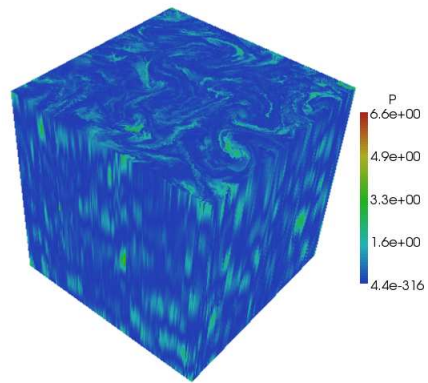
(a)



(b)



(c)



(d)

